

**BLOW-UP PHENOMENA OF ARBITRARY POSITIVE
INITIAL ENERGY SOLUTIONS FOR A VISCOELASTIC
WAVE EQUATION WITH NONLINEAR DAMPING
AND SOURCE TERMS**

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ABSTRACT. In this paper, we considered the Dirichlet initial boundary value problem of a nonlinear viscoelastic wave equation with nonlinear damping and source terms, and investigated finite time blow-up phenomena of the solutions to the equation with arbitrary positive initial data, under suitable conditions.

1. Introduction

Consider the viscoelastic wave equation with nonlinear damping and source terms

$$(1.1) \quad \begin{aligned} & u_{tt} - \Delta u_t - \operatorname{div}(|\nabla u|^{\alpha-2} \nabla u) - \operatorname{div}(|\nabla u_t|^{\beta-2} \nabla u_t) + |u_t|^{m-2} u_t \\ &= |u|^{p-2} u, \quad (x, t) \in \Omega \times (0, \infty), \end{aligned}$$

under the homogeneous Dirichlet boundary and initial conditions

$$(1.2) \quad u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty),$$

$$(1.3) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$, and $\alpha, \beta, p, m > 2$ are constants and $u_0, u_1 : \Omega \rightarrow \mathbb{R}$ are given initial data.

The equation (1.1) appears in the models of nonlinear viscoelasticity such as the system governing the longitudinal motion of a viscoelastic

Received February 25, 2022; Accepted May 01, 2022.

2010 Mathematics Subject Classification: 35L05, 35B40, 46E05.

Key words and phrases: viscoelastic wave equation, blow-up, arbitrary positive initial energy.

This research was supported by Changwon National University in 2021–2022.

configuration obeying a nonlinear Voight model, see [1] and [3]. The interaction between the damping and source terms was firstly investigated by Levine, considering the semilinear wave equation

$$(1.4) \quad u_{tt} - \Delta u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad (x, t) \in \Omega \times (0, \infty),$$

where $m = 2$, and the author showed that the solutions with negative initial energy blow up in finite time (cf. [4], [5]). Later, Georgiev and Todorova [2] extended Levine's results to the nonlinear case, i.e., $m > 2$, and they introduced a different method and showed that global solutions of (1.4) exist if $2 < p \leq m$ and the initial energy is negative, and that the solution blows up in finite time if $2 < m < p$ and the initial energy is sufficiently negative. Vitillaro [9] considered an abstract equation, including linear and quasilinear cases, with nonlinear damping term, and presented a blow-up result of the solutions with small positive energy. Messaoudi [6] showed that any solution of (1.4) with negative initial energy blows up in finite time if $m < p$, whereas Georgiev and Todorova [2] showed that the solutions of (1.4) blow up in finite time if the initial energy is sufficiently negative.

Recently, Yang [10] studied problem (1.1)-(1.3) with $p > \max\{\alpha, m\}$ and $\alpha > \beta$, and established a blow-up result, when the initial energy is sufficiently negative. Afterwards, Messaoudi and Houari [7] extended the result to the case that the initial energy is negative. Besides, on the blow-up of solutions with arbitrary positive initial energy for other type wave equations, one can refer to [8] and [11].

In this paper, inspired by [7] and [8], we establish a blow-up result for the solutions to problem (1.1)-(1.3) with arbitrary positive initial energy. We first introduce the following function space, energy functional, and lemma:

$$(1.5) \quad \begin{aligned} Z &= L^\infty([0, T]; W_0^{1,\alpha}(\Omega)) \cap W^{1,\infty}([0, T]; L^2(\Omega)) \\ &\quad \cap W^{1,\beta}([0, T]; W_0^{1,\beta}(\Omega)) \cap W^{1,m}([0, T]; L^m(\Omega)), \\ E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha - \frac{1}{p}\|u\|_p^p, \quad t > 0, \end{aligned}$$

LEMMA 1.1. *Suppose that $\alpha, \beta, p, m > 2$, $\alpha > \beta$, and $\max\{m, \alpha\} < p < r_\alpha$, where r_α is the Sobolev critical exponent of $W_0^{1,\alpha}(\Omega)$. If there exists a number $t_0 \geq 0$ such that $E(t_0) < 0$, then the solution $u \in Z$ of problem (1.1)-(1.3) blows up in finite time.*

REMARK 1.2. We remind that $r_\alpha = \frac{N\alpha}{N-\alpha}$, if $N > \alpha$, and $r_\alpha > \alpha$, if $N = \alpha$. In addition, $r_\alpha = \infty$, if $N < \alpha$.

Lemma 1.1 can be easily proved by a similar argument as in [7] with slight modification, and hence, we omit the proof here.

2. Main result and an example

THEOREM 2.1. *Suppose that $\alpha, \beta, p, m > 2, \alpha > \beta$, and $\max\{m, \alpha\} < p < r_\alpha$. If $u(t) \in Z$ is a solution of problem (1.1)-(1.3) satisfying*

$$(2.1) \quad E(0) > 0 \text{ and } \int_{\Omega} u_0 u_1 dx > ME(0),$$

then $u(t)$ blows up in finite time, where

$$\begin{aligned} M &= \eta(\varepsilon_0), \\ \eta(\varepsilon) &= C_2 \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right]^{-1} + \frac{\beta-1}{\beta} \left\{ \frac{\beta}{4C_1} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] \right\}^{-\frac{1}{\beta-1}} \\ &\quad + \frac{m-1}{m} \left(\frac{p\varepsilon}{1-\theta} \right)^{-\frac{1}{m-1}}, \\ \theta &= \frac{p-m}{p-2}, \end{aligned}$$

and $\varepsilon_0 \in (0, 1)$ is the root of the equation $\frac{p(1-\varepsilon_0)}{\gamma(\varepsilon_0)} = \eta(\varepsilon_0)$,

$$\gamma(\varepsilon) = 2\sqrt{\left[1 + \frac{p(1-\varepsilon)}{2} \right] \left\{ \frac{\lambda_1}{2C_2} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] - \frac{p\theta\varepsilon}{2(1-\theta)} \right\}},$$

$\lambda_1 > 0$ is the first eigenvalue of $-\Delta$, and C_1 and C_2 are positive constants.

Proof. By contradiction, we assume that $u(t) \in Z$ is a global solution of problem (1.1)-(1.3). Multiplying the both sides of equation (1.1) by u_t and integrating the result over Ω , one can see that

$$(2.2) \quad \frac{d}{dt} E(t) = -\|\nabla u_t\|_2^2 - \|\nabla u_t\|_\beta^\beta - \|u_t\|_m^m \leq 0.$$

It follows from Green's formula that

$$\begin{aligned} (2.3) \quad \frac{d}{dt} \int_{\Omega} u u_t dx &= \|u_t\|_2^2 + \int_{\Omega} u u_{tt} dx \\ &= \|u_t\|_2^2 - \|\nabla u\|_\alpha^\alpha - \int_{\Omega} |\nabla u_t|^{\beta-2} \nabla u_t \cdot \nabla u dx \\ &\quad - \int_{\Omega} \nabla u_t \cdot \nabla u dx - \int_{\Omega} |u_t|^{m-2} u_t u dx + \|u\|_p^p, \end{aligned}$$

by multiplying the both sides of (1.1) by u and integrating the result over Ω .

In order to estimate the right side of (2.3), we use Hölder's inequality and Young's inequality and obtain the inequalities

$$\begin{aligned}
(2.4) \quad & \left| \int_{\Omega} \nabla u_t \cdot \nabla u dx \right| \leq \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u_t|^2 dx \right)^{\frac{1}{2}} \\
& = \left[(2\mu)^{\frac{1}{2}} \|\nabla u\|_2 \right] \left[\frac{1}{(2\mu)^{\frac{1}{2}}} \|\nabla u_t\|_2 \right] \\
& \leq \frac{1}{2} \left[(2\mu)^{\frac{1}{2}} \|\nabla u\|_2 \right]^2 + \frac{1}{2} \left[\frac{1}{(2\mu)^{\frac{1}{2}}} \|\nabla u_t\|_2 \right]^2 \\
& = \mu \|\nabla u\|_2^2 + \frac{1}{4\mu} \|\nabla u_t\|_2^2,
\end{aligned}$$

$$\begin{aligned}
(2.5) \quad & \left| \int_{\Omega} |\nabla u_t|^{\beta-2} \nabla u_t \cdot \nabla u dx \right| \leq \left(\int_{\Omega} |\nabla u|^{\beta} dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega} |\nabla u_t|^{\beta} dx \right)^{\frac{\beta-1}{\beta}} \\
& = (\lambda \|\nabla u\|_{\beta}) \left(\lambda^{-1} \|\nabla u_t\|_{\beta}^{\beta-1} \right) \\
& \leq \frac{1}{\beta} (\lambda \|\nabla u\|_{\beta})^{\beta} + \frac{\beta-1}{\beta} \left(\lambda^{-1} \|\nabla u_t\|_{\beta}^{\beta-1} \right)^{\frac{\beta}{\beta-1}} \\
& = \frac{\lambda^{\beta}}{\beta} \|\nabla u\|_{\beta}^{\beta} + \frac{\beta-1}{\beta} \lambda^{-\frac{\beta}{\beta-1}} \|\nabla u_t\|_{\beta}^{\beta},
\end{aligned}$$

and

$$\begin{aligned}
(2.6) \quad & \left| \int_{\Omega} |u_t|^{m-2} u_t u dx \right| \leq \left(\int_{\Omega} |u|^m dx \right)^{\frac{1}{m}} \left(\int_{\Omega} |u_t|^m dx \right)^{\frac{m-1}{m}} \\
& = (\delta \|u\|_m) (\delta^{-1} \|u_t\|_m^{m-1}) \\
& \leq \frac{1}{m} (\delta \|u\|_m)^m + \frac{m-1}{m} (\delta^{-1} \|u_t\|_m^{m-1})^{\frac{m}{m-1}} \\
& = \frac{\delta^m}{m} \|u\|_m^m + \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \|u_t\|_m^m,
\end{aligned}$$

where $\mu, \lambda, \delta > 0$ are constants which will be specified later. Substituting inequalities (2.4)-(2.6) into (2.3), one can have the inequality

$$(2.7) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} uu_t \, dx &\geq \|u_t\|_2^2 - \|\nabla u\|_{\alpha}^{\alpha} - \mu \|\nabla u\|_2^2 - \frac{1}{4\mu} \|\nabla u_t\|_2^2 \\ &\quad - \frac{\lambda^{\beta}}{\beta} \|\nabla u\|_{\beta}^{\beta} - \frac{\beta-1}{\beta} \lambda^{-\frac{\beta}{\beta-1}} \|\nabla u_t\|_{\beta}^{\beta} - \frac{\delta^m}{m} \|u\|_m^m \\ &\quad - \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \|u_t\|_m^m + \|u\|_p^p. \end{aligned}$$

On the other hand, it follows from (1.5) that

$$\|u\|_p^p = -pE(t) + \frac{p}{2} \|u_t\|_2^2 + \frac{p}{\alpha} \|\nabla u\|_{\alpha}^{\alpha},$$

and from (2.2) one can see that

$$-\|\nabla u_t\|_2^2 \geq \frac{d}{dt} E(t),$$

$$-\|\nabla u_t\|_{\beta}^{\beta} \geq \frac{d}{dt} E(t),$$

and

$$-\|u_t\|_m^m \geq \frac{d}{dt} E(t).$$

Then inequality (2.7) can be written as

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} uu_t \, dx &\geq \|u_t\|_2^2 - \|\nabla u\|_{\alpha}^{\alpha} - \mu \|\nabla u\|_2^2 - \frac{1}{4\mu} \|\nabla u_t\|_2^2 \\ &\quad - \frac{\lambda^{\beta}}{\beta} \|\nabla u\|_{\beta}^{\beta} - \frac{\beta-1}{\beta} \lambda^{-\frac{\beta}{\beta-1}} \|\nabla u_t\|_{\beta}^{\beta} - \frac{\delta^m}{m} \|u\|_m^m \\ &\quad - \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \|u_t\|_m^m + \varepsilon \|u\|_p^p + (1-\varepsilon) \|u\|_p^p \\ &\geq \left(\frac{1}{4\mu} + \frac{\beta-1}{\beta} \lambda^{-\frac{\beta}{\beta-1}} + \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \right) \frac{d}{dt} E(t) \\ &\quad + \left[1 + \frac{p(1-\varepsilon)}{2} \right] \|u_t\|_2^2 + \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] \|\nabla u\|_{\alpha}^{\alpha} \\ &\quad - \mu \|\nabla u\|_2^2 - \frac{\lambda^{\beta}}{\beta} \|\nabla u\|_{\beta}^{\beta} - \frac{\delta^m}{m} \|u\|_m^m + \varepsilon \|u\|_p^p - p(1-\varepsilon)E(t), \end{aligned}$$

where ε is a constant small enough, from which we can have the inequality

$$(2.8) \quad \begin{aligned} & \frac{d}{dt} \left[\int_{\Omega} uu_t dx - \left(\frac{1}{4\mu} + \frac{\beta-1}{\beta} \lambda^{-\frac{\beta}{\beta-1}} + \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \right) E(t) \right] \\ & \geq \left[1 + \frac{p(1-\varepsilon)}{2} \right] \|u_t\|_2^2 + \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] \|\nabla u\|_{\alpha}^{\alpha} - \mu \|\nabla u\|_2^2 \\ & \quad - \frac{\lambda^{\beta}}{\beta} \|\nabla u\|_{\beta}^{\beta} - \frac{\delta^m}{m} \|u\|_m^m + \varepsilon \|u\|_p^p - p(1-\varepsilon)E(t). \end{aligned}$$

Using Hölder's inequality, one can see that there exists a constant $C_1 > 0$ such that

$$(2.9) \quad \|\nabla u\|_{\beta}^{\beta} \leq |\Omega|^{1-\frac{\beta}{\alpha}} \|\nabla u\|_{\alpha}^{\beta} \leq C_1 \|\nabla u\|_{\alpha}^{\alpha}.$$

In fact, we have $\|\nabla u\|_{\alpha}^{\beta} \leq \|\nabla u\|_{\alpha}^{\alpha}$, if $\|\nabla u\|_{\alpha} \geq 1$, and there exists a constant $C > 0$ such that $\|\nabla u\|_{\alpha}^{\beta} \leq C \|\nabla u\|_{\alpha}^{\alpha}$, if $0 < \|\nabla u\|_{\alpha} < 1$. The inequality (2.9) clearly holds, if $\|\nabla u\|_{\alpha} = 0$. Similarly, it can be seen that there exists a constant $C_2 > 0$ such that

$$\|\nabla u\|_2^2 \leq C_2 \|\nabla u\|_{\alpha}^{\alpha}.$$

Then, from (2.8), we have the inequality

$$(2.10) \quad \begin{aligned} & \frac{d}{dt} \left[\int_{\Omega} uu_t dx - \left(\frac{1}{4\mu} + \frac{\beta-1}{\beta} \lambda^{-\frac{\beta}{\beta-1}} + \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \right) E(t) \right] \\ & \geq \left[1 + \frac{p(1-\varepsilon)}{2} \right] \|u_t\|_2^2 + \left[\frac{p(1-\varepsilon)}{\alpha} - 1 - \frac{\lambda^{\beta} C_1}{\beta} - \mu C_2 \right] \|\nabla u\|_{\alpha}^{\alpha} \\ & \quad - \frac{\delta^m}{m} \|u\|_m^m + \varepsilon \|u\|_p^p - p(1-\varepsilon)E(t). \end{aligned}$$

Meanwhile, from the convexity of the function $\frac{u^y}{y}$ in y for $u \geq 0$ and $y > 0$, we have the inequality

$$(2.11) \quad \frac{1}{m} \|u\|_m^m \leq \frac{\theta}{2} \|u\|_2^2 + \frac{1-\theta}{p} \|u\|_p^p,$$

where $\theta = \frac{p-m}{p-2}$. It then follows from inequalities (2.10) and (2.11) that

$$\begin{aligned}
 & \frac{d}{dt} \left[\int_{\Omega} uu_t dx - \left(\frac{1}{4\mu} + \frac{\beta-1}{\beta} \lambda^{-\frac{\beta}{\beta-1}} + \frac{m-1}{m} \delta^{-\frac{m}{m-1}} \right) E(t) \right] \\
 (2.12) \quad & \geq \left[1 + \frac{p(1-\varepsilon)}{2} \right] \|u_t\|_2^2 + \left[\frac{p(1-\varepsilon)}{\alpha} - 1 - \frac{\lambda^\beta C_1}{\beta} - \mu C_2 \right] \|\nabla u\|_\alpha^\alpha \\
 & \quad - \frac{\theta \delta^m}{2} \|u\|_2^2 + \left[\varepsilon - \frac{(1-\theta)\delta^m}{p} \right] \|u\|_p^p - p(1-\varepsilon)E(t).
 \end{aligned}$$

Setting $\delta = \left(\frac{\varepsilon p}{1-\theta} \right)^{\frac{1}{m}}$, $\lambda = \left\{ \frac{\beta}{4C_1} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] \right\}^{\frac{1}{\beta}}$, and $\mu = \frac{1}{4C_2} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right]$ in (2.12), and using Poincaré’s inequality, inequality (2.12) becomes

$$\begin{aligned}
 (2.13) \quad & \frac{d}{dt} \left[\int_{\Omega} uu_t dx - \eta(\varepsilon)E(t) \right] \\
 & \geq \left[1 + \frac{p(1-\varepsilon)}{2} \right] \|u_t\|_2^2 + \left\{ \frac{\lambda_1}{2C_2} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] - \frac{p\theta\varepsilon}{2(1-\theta)} \right\} \|u\|_2^2 \\
 & \quad - p(1-\varepsilon)E(t),
 \end{aligned}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$.

Let ε be a constant small enough such that $0 < \varepsilon < 1$ and

$$\frac{\lambda_1}{2C_2} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] - \frac{p\theta\varepsilon}{2(1-\theta)} > 0.$$

We then have the inequality

$$\begin{aligned}
 & \left[1 + \frac{p(1-\varepsilon)}{2} \right] \|u_t\|_2^2 + \left\{ \frac{\lambda_1}{2C_2} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] - \frac{p\theta\varepsilon}{2(1-\theta)} \right\} \|u\|_2^2 \\
 (2.14) \quad & \geq 2\sqrt{\left[1 + \frac{p(1-\varepsilon)}{2} \right] \left\{ \frac{\lambda_1}{2C_2} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] - \frac{p\theta\varepsilon}{2(1-\theta)} \right\}} \int_{\Omega} uu_t dx,
 \end{aligned}$$

by Cauchy’s inequality, and it follows from (2.13) and (2.14) that

$$\begin{aligned}
 (2.15) \quad & \frac{d}{dt} \left[\int_{\Omega} uu_t dx - \eta(\varepsilon)E(t) \right] \geq \gamma(\varepsilon) \int_{\Omega} uu_t dx - p(1-\varepsilon)E(t) \\
 & = \gamma(\varepsilon) \left[\int_{\Omega} uu_t dx - \frac{p(1-\varepsilon)}{\gamma(\varepsilon)} E(t) \right].
 \end{aligned}$$

It can be easily seen that

$$\begin{aligned}
& \left[1 + \frac{p(1-\varepsilon)}{2} \right] \left\{ \frac{\lambda_1}{2C_2} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] - \frac{p\theta\varepsilon}{2(1-\theta)} \right\} \\
& \rightarrow \frac{\lambda_1}{2C_2} \left(1 + \frac{p}{2} \right) \left(\frac{p}{\alpha} - 1 \right), \\
(2.16) \quad \gamma(\varepsilon) & \rightarrow \sqrt{\frac{2\lambda_1}{C_2} \left(1 + \frac{p}{2} \right) \left(\frac{p}{\alpha} - 1 \right)}, \\
\frac{p(1-\varepsilon)}{\gamma(\varepsilon)} & \rightarrow \frac{p}{\sqrt{\frac{2\lambda_1}{C_2} \left(1 + \frac{p}{2} \right) \left(\frac{p}{\alpha} - 1 \right)}}, \quad \eta(\varepsilon) \rightarrow \infty, \text{ as } \varepsilon \rightarrow 0^+,
\end{aligned}$$

and

$$\left[1 + \frac{p(1-\varepsilon)}{2} \right] \left\{ \frac{\lambda_1}{2C_2} \left[\frac{p(1-\varepsilon)}{\alpha} - 1 \right] - \frac{p\theta\varepsilon}{2(1-\theta)} \right\} \rightarrow - \left[\frac{\lambda_1}{2C_2} + \frac{p\theta}{2(1-\theta)} \right],$$

as $\varepsilon \rightarrow 1^-$. Hence, there exists a constant $\varepsilon_* \in (0, 1)$ such that

$$\gamma(\varepsilon_*) = 0 \text{ and } \gamma(\varepsilon) > 0 \text{ for all } \varepsilon \in (0, \varepsilon_*),$$

which implies that

$$\begin{aligned}
(2.17) \quad \gamma(\varepsilon) & \rightarrow 0, \quad \frac{p(1-\varepsilon)}{\gamma(\varepsilon)} \rightarrow \infty, \\
\eta(\varepsilon) & \rightarrow C_2 \left[\frac{p(1-\varepsilon_*)}{\alpha} - 1 \right]^{-1} + \frac{\beta-1}{\beta} \left\{ \frac{\beta}{4C_1} \left[\frac{p(1-\varepsilon_*)}{\alpha} - 1 \right] \right\}^{-\frac{1}{\beta-1}} \\
& \quad + \frac{m-1}{m} \left(\frac{p\varepsilon_*}{1-\theta} \right)^{-\frac{1}{m-1}},
\end{aligned}$$

as $\varepsilon \rightarrow \varepsilon_*^-$.

By virtue of (2.16), (2.17), and the continuity of $\frac{p(1-\varepsilon)}{\gamma(\varepsilon)}$ in $\varepsilon \in (0, \varepsilon_*)$, there exists a constant $\varepsilon_0 \in (0, \varepsilon_*) \subset (0, 1)$ such that

$$\frac{p(1-\varepsilon_0)}{\gamma(\varepsilon_0)} = \eta(\varepsilon_0).$$

From (2.15), one can obtain the inequality

$$(2.18) \quad \frac{d}{dt} \left[\int_{\Omega} uu_t \, dx - \eta(\varepsilon_0)E(t) \right] \geq \gamma(\varepsilon_0) \left[\int_{\Omega} uu_t \, dx - \eta(\varepsilon_0)E(t) \right].$$

We now define a function $H : [0, \infty) \rightarrow R$ as

$$H(t) = \int_{\Omega} uu_t \, dx - \eta(\varepsilon_0)E(t).$$

Then, from condition (2.1) and (2.18), we have

$$H(0) = \int_{\Omega} u_0u_1 \, dx - \eta(\varepsilon_0)E(0) > 0,$$

and

$$(2.19) \quad \frac{d}{dt}H(t) \geq \gamma(\varepsilon_0)H(t),$$

and inequality (2.19) implies that

$$H(t) \geq e^{\gamma(\varepsilon_0)t}H(0) \text{ for all } t \geq 0.$$

Since $u(t)$ is a global solution, one can see that $0 \leq E(t) \leq E(0)$ for all $t \geq 0$ by Lemma 1.1 and (2.2), and hence, we have the inequalities

$$\int_{\Omega} uu_t \, dx \geq H(t) \geq e^{\gamma(\varepsilon_0)t}H(0).$$

Therefore, we obtain the estimates

$$(2.20) \quad \begin{aligned} \|u(t)\|_2^2 &= \|u(0)\|_2^2 + 2 \int_0^t \int_{\Omega} uu_{\tau} \, dx d\tau \\ &\geq \|u(0)\|_2^2 + 2 \int_0^t e^{\gamma(\varepsilon_0)\tau}H(0) \, d\tau \\ &= \|u(0)\|_2^2 + \frac{2}{\gamma(\varepsilon_0)}(e^{\gamma(\varepsilon_0)t} - 1)H(0). \end{aligned}$$

On the other hand, by Hölder's inequality and (2.2), we have

$$(2.21) \quad \begin{aligned} \|u(t)\|_2 &\leq \|u(0)\|_2 + \int_0^t \|u_{\tau}\|_2 \, d\tau \\ &\leq \|u(0)\|_2 + C_0 \int_0^t \|u_{\tau}\|_m \, d\tau \\ &\leq \|u(0)\|_2 + C_0 t^{\frac{m-1}{m}} \left(\int_0^t \|u_{\tau}\|_m^m \, d\tau \right)^{\frac{1}{m}} \\ &\leq \|u(0)\|_2 + C_0 t^{\frac{m-1}{m}} [E(0) - E(t)]^{\frac{1}{m}} \\ &\leq \|u(0)\|_2 + C_0 t^{\frac{m-1}{m}} E(0)^{\frac{1}{m}}, \end{aligned}$$

where C_0 is a positive constant, which contradicts with (2.20). The proof is completed. \square

EXAMPLE 2.2. As an application of Theorem 2.1, we consider the following example in one-dimensional space:

Let $\Omega = [0, 2\pi] \subset \mathbb{R}$ and assume that $\beta < \alpha = 4$ and $m < p = 5$. The initial data u_0 and u_1 are given by

$$u(x, 0) = a \sin(bx) \text{ and } u_t(x, 0) = a^2 b^2 \sin(bx),$$

where a and b are positive integers.

Then one can see that

$$\begin{aligned} E(0) &= \frac{1}{2} \|u_t(0)\|_2^2 + \frac{1}{4} \|\nabla u(0)\|_4^4 - \frac{1}{5} \|u(0)\|_5^5 \\ &= \frac{1}{2} \int_0^{2\pi} |a^2 b^2 \sin(bx)|^2 dx + \frac{1}{4} \int_0^{2\pi} |ab \cos(bx)|^4 dx \\ &\quad - \frac{1}{5} \int_0^{2\pi} |a \sin(bx)|^5 dx \\ &= \frac{5}{8} a^4 b^4 \pi - \frac{32}{75} a^5, \end{aligned} \tag{2.22}$$

and

$$\int_{\Omega} u(0)u_t(0) dx = a^3 b^2 \pi.$$

It can be easily seen that for any given constant $c > 0$, there exist constants $a, b > 0$ such that $\frac{a^3 b^2 \pi}{M} > c$ and

$$\frac{5}{8} a^4 b^4 \pi - \frac{32}{75} a^5 - \frac{a^3 b^2 \pi}{2M} = 0,$$

where M is a constant defined in (2.1), which implies

$$\int_{\Omega} u_0 u_1 dx > M E(0).$$

By Theorem 2.1, we conclude that the solution of problem (1.1)-(1.3) blows up in finite time.

Acknowledgments

The author would like to deeply thank all the reviewers for their insightful and constructive comments.

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